

# Absolute gradings on ECH and Heegaard Floer homology

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## Abstract

In joint work with Yang Huang, we defined a canonical absolute grading on Heegaard Floer homology by homotopy classes of oriented 2-plane fields. A similar grading was defined on embedded contact homology by Michael Hutchings. In this paper we show that the isomorphism between these homology theories defined by Colin-Ghiggini-Honda preserves this grading.

## 1 Introduction

Let  $Y$  be a closed, connected, oriented three-manifold. The embedded contact homology (ECH) and the Heegaard Floer homology of  $Y$  are invariants that have been studied and computed for many manifolds. ECH was defined by Hutchings using a contact form on  $Y$ , see [7, 9], and Heegaard Floer homology was defined in [15] by Ozsváth-Szabó using a Heegaard decomposition of  $Y$ . These two homology theories have very distinct flavors, but they have recently been shown to be isomorphic by Colin-Ghiggini-Honda [1, 2, 3]. More specifically, they construct an isomorphism  $\Phi : HF^+(-Y) \rightarrow ECH(Y)$ . Here  $HF^+(-Y)$  is a version of Heegaard Floer homology of  $Y$  with the opposite orientation. One could instead consider the Heegaard Floer cohomology of  $Y$  with the right orientation.

One can define a canonical absolute grading on ECH and Heegaard Floer homology by homotopy classes of oriented 2-plane fields on  $Y$ . We now

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explain why this is a natural grading set to consider. First note that the orientation of  $Y$  induces a bijection between the homotopy classes of oriented 2-plane fields and the homotopy classes of nonvanishing vector fields. Let  $\text{Vect}(Y)$  denote the set of homotopy classes of nonvanishing vector fields on  $Y$ . One also defines  $\text{Spin}^c(Y)$  to be the set of equivalence classes of vector fields on  $Y$ , where two vector fields are said to be equivalent if they are homotopic in the complement of an embedded three-dimensional ball. Let  $\pi : \text{Vect}(Y) \rightarrow \text{Spin}^c(Y)$  denote the quotient map. We observe that  $\mathbb{Z}$  acts on  $\text{Vect}(Y)$  as follows. Let  $[v] \in \text{Vect}(Y)$  and let  $B$  be a small embedded ball in  $Y$ . Fix a trivialization of  $TY$ . So we can see  $v$  as a map  $v : Y \rightarrow S^2$ . Without loss of generality, we can assume that  $v$  is constant in  $B$ . Let  $v'$  be the vector field given by the composition of the quotient map  $B \rightarrow B/\partial B \cong S^3$  with the Hopf map  $S^3 \rightarrow S^2$ . One can check that the homotopy class  $[v']$  does not depend on the choice of  $B$  or  $v$ . So  $[v] + 1$  is defined to be  $[v']$ . By induction, one defines an action of  $\mathbb{Z}$  on  $\text{Vect}(Y)$ . It follows from the definition that  $\pi$  is invariant under this action. Moreover, one can check that for  $\mathfrak{s} \in \text{Spin}^c(Y)$ , any two elements in  $\pi^{-1}(\mathfrak{s})$  differ by  $n$ , for some  $n \in \mathbb{Z}$ . In fact, the set  $\pi^{-1}(\mathfrak{s})$  is an affine space over  $\mathbb{Z}/d$ , where  $d$  is the divisibility of  $c_1(\mathfrak{s}) \in H^2(Y; \mathbb{Z})$ . One can prove all these facts using the Pontryagin-Thom construction, for example. Now, since all versions of ECH and Heegaard Floer homology split according to  $\text{spin}^c$  structures and they are relatively graded on  $\mathbb{Z}/d$  for the appropriate  $d$  for each  $\text{spin}^c$  structure, the set  $\text{Vect}(Y)$  is a possible choice for an absolute grading set. In fact, absolute gradings with values on  $\text{Vect}(Y)$  were defined without any extra choices on ECH and Heegaard Floer homology [8, 6].

For  $j \in \text{Vect}(Y)$ , let  $HF_\rho^+(-Y)$  and  $ECH_\rho(Y)$  denote the subgroups of  $HF^+(-Y)$  and  $ECH(Y)$ , respectively, consisting of all elements of grading  $\rho \in \text{Vect}(Y)$ .

**Theorem 1.1.** *Let  $\Phi : HF^+(-Y) \rightarrow ECH(Y)$  be the isomorphism constructed by Colin-Ghiggini-Honda. Then  $\Phi$  maps  $HF_\rho^+(-Y)$  to  $ECH_\rho(Y)$  for all  $\rho \in \text{Vect}(Y)$ .*

We recall that both  $HF^+(-Y)$  and  $ECH(Y)$  admit a map  $U$  whose mapping cone is denoted by  $\widehat{HF}(-Y)$  and  $\widehat{ECH}(Y)$ , respectively. The latter groups are simpler than the former and they are also isomorphic. In fact, in order to show that  $\Phi$  is an isomorphism, Colin, Ghiggini and Honda first construct an isomorphism  $\widehat{\Phi} : \widehat{HF}(-Y) \rightarrow \widehat{ECH}(Y)$ . They also show that

the following diagram commutes.

$$\begin{array}{ccc}
\widehat{HF}(-Y) & \xrightarrow{\iota_*} & HF(-Y) \\
\downarrow \widehat{\Phi} & & \downarrow \Phi \\
\widehat{ECH}(Y) & \xrightarrow{\iota_*} & ECH(Y)
\end{array} \tag{1}$$

Here the horizontal maps  $\iota_*$  denote the natural maps given by the mapping cone construction. In order to show that  $\Phi$  preserves the absolute grading, it is enough to prove that both maps  $\iota_*$  and  $\widehat{\Phi}$  do.

The map  $\widehat{\Phi}$  is defined as a composition  $\widehat{\Phi} = \psi \circ \widetilde{\Phi} \circ \psi'$  as follows.

$$\widehat{HF}(-Y) \xrightarrow{\psi'} \widehat{HF}(S, \mathbf{a}, \varphi(\mathbf{a})) \xrightarrow{\widetilde{\Phi}} ECH_{2g}(N_{(S, \varphi)}, \lambda) \xrightarrow{\psi} \widehat{ECH}(Y).$$

Here  $\widehat{HF}(S, \mathbf{a}, \varphi(\mathbf{a}))$  is the homology of a chain group computed from the page of an open book decomposition  $(S, \varphi)$  of  $Y$  and  $ECH_{2g}(N_{(S, \varphi)}, \lambda)$  is the homology of a subcomplex of the ECH chain complex  $ECC(Y, \lambda)$  generated by orbits sets that intersect  $S$  exactly  $2g$  times, where  $\lambda$  is a certain contact form on  $N$ . The maps  $\psi'$ ,  $\widetilde{\Phi}$  and  $\psi$  are all isomorphisms and we will show that all of them preserve the grading.

This paper is organized as follows. In Section 2, we review the definition Heegaard Floer homology, using Lipshitz's cylindrical reformulation, and the absolute grading on it. We explain how to compute its “hat”-version from a page of an open book decomposition, following [4]. We also prove that  $\psi'$  preserves the grading. In Section 3, we recall the definition of ECH and its absolute grading. We explain its “hat”-version and the relationship between ECH and open book decompositions. We show how to construct the map  $\psi$  and we prove that it preserves the grading. In Section 4, we recall some of the steps to construct the isomorphism  $\widetilde{\Phi}$  and we finish the proof of Theorem 1.1.

## 2 The grading on Heegaard Floer homology

In this section, we will recall the definition of Heegaard Floer homology using the cylindrical reformulation from [14] and the absolute grading on it from [6].

### 2.1 Heegaard Floer homology

A Heegaard diagram is a triple  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ , where  $\Sigma$  is a closed oriented surface of genus  $g$  and the tuples  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g)$  and  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_g)$  are  $g$ -tuples

of disjoint circles on  $\Sigma$  which are linearly independent in  $H_1(\Sigma)$ . Given such a Heegaard diagram, one can construct a closed oriented three-manifold by considering  $\Sigma \times [0, 1]$ , attaching disks to it along the circles  $\alpha_i \times \{0\}$  and  $\beta_j \times \{1\}$  and filling the rest with two three-dimensional balls. Any closed oriented three-manifold  $Y$  can be obtained this way from a Heegaard diagram.

A pointed Heegaard diagram is a quadruple  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$  where  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$  is a Heegaard diagram and  $z$  is a point on  $\Sigma$  in the complement of all of the circles  $\alpha_i$  and  $\beta_j$ . Given a pointed Heegaard diagram  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ , an intersection point is a  $g$ -tuple  $\mathbf{x} = (x_1, \dots, x_g)$ , where  $x_i \in \alpha_i \cap \beta_{\varphi(i)}$  and  $\varphi$  is a permutation of  $\{1, \dots, g\}$ . The chain complex  $\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$  is the  $\mathbb{Z}$ -module generated by the intersection points. The differential  $\partial$  is defined as follows. Consider the manifold  $W = \mathbb{R} \times [0, 1] \times \Sigma$  with the symplectic form  $ds \wedge dt + \omega$ , where  $s$  and  $t$  are the coordinates on the  $\mathbb{R}$ - and  $[0, 1]$ -factors, respectively, and  $\omega$  is an area form on  $\Sigma$ . If  $\mathbf{x} = (x_1, \dots, x_g)$  and  $\mathbf{y} = (y_1, \dots, y_g)$ , one defines a moduli space  $\mathcal{M}(\mathbf{x}, \mathbf{y})$ , which is basically the moduli space of holomorphic curves in  $W$  with boundary on the union of the cylinders  $\mathbb{R} \times \{1\} \times \alpha_i$  and  $\mathbb{R} \times \{0\} \times \beta_j$ , and that converge to the union of chords  $[0, 1] \times x_i$  as  $s \rightarrow \infty$  and to the union of  $[0, 1] \times y_j$  as  $s \rightarrow -\infty$ . For details, see [14]. Now  $\langle \partial \mathbf{x}, \mathbf{y} \rangle$  is defined to be the signed count of certain embedded holomorphic curves in  $\mathcal{M}(\mathbf{x}, \mathbf{y})$  that do not intersect  $\mathbb{R} \times [0, 1] \times \{z\}$ . It turns out that  $\partial^2 = 0$ , so one defines  $\widehat{HF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$  to be the homology of this chain complex.

We now recall the definition of the other versions of Heegaard Floer homology. The complex  $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$  is defined to be the  $\mathbb{Z}$ -module generated by  $[\mathbf{x}, n]$ , where  $\mathbf{x}$  is an intersection point and  $n \in \mathbb{Z}$ . The differential is defined by

$$\partial[\mathbf{x}, n] = \sum_{\mathbf{y}, A} c(\mathbf{x}, \mathbf{y}, A)[\mathbf{y}, n - n_z(A)],$$

where  $A$  is a relative homology class in  $W$ , the number  $c(\mathbf{x}, \mathbf{y}, A)$  is the signed count of certain holomorphic curves in  $\mathbf{x}, \mathbf{y}$  in  $A$  and  $n_z(A)$  is the intersection number of  $A$  and  $\mathbb{R} \times [0, 1] \times \{z\}$ . One can show that  $\partial^2 = 0$ . One can now define  $CF^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$  to be the subgroup of  $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$  generated by  $[\mathbf{x}, n]$ , for  $n < 0$ . This is a subcomplex because  $n_z(A) \geq 0$ , whenever  $A$  can be represented by a holomorphic curve. One also defines  $CF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$  to be the quotient of  $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$  by  $CF^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ . The homologies of

these various complexes are denoted, respectively, by

$$HF^\infty(\Sigma, \alpha, \beta, z), HF^-(\Sigma, \alpha, \beta, z) \text{ and } HF^+(\Sigma, \alpha, \beta, z).$$

The  $U$  map is defined by  $U[\mathbf{x}, m] = [\mathbf{x}, m - 1]$ . So there is a short exact sequence

$$0 \rightarrow \widehat{CF}(\Sigma, \alpha, \beta, z) \hookrightarrow CF^+(\Sigma, \alpha, \beta, z) \xrightarrow{U} CF^+(\Sigma, \alpha, \beta, z) \rightarrow 0.$$

This sequence induces an exact triangle.

$$\begin{array}{ccc} HF^+(\Sigma, \alpha, \beta, z) & \xrightarrow{U} & HF^+(\Sigma, \alpha, \beta, z) \\ & \nwarrow \quad \nearrow & \\ & \widehat{HF}(\Sigma, \alpha, \beta, z) & \end{array} \quad (2)$$

## 2.2 The absolute grading

We will now recall the absolute grading on all these homology groups. Let  $(f, V)$  be a pair consisting of a self-indexing Morse function  $f$  on  $Y$  and a gradient-like vector field  $V$ , i.e.  $df(V) > 0$ , whenever  $df \neq 0$ . We assume that  $f$  has exactly one index 0 and one index 3 critical points. We also assume that all stable and unstable manifolds intersect transversely. For each index 1 critical point  $p_i$ , let  $U_i$  denote the unstable manifold containing  $p_i$  and, for each index 2 critical point  $q_j$ , let  $S_j$  denote the stable manifold containing  $q_j$ . The pair  $(f, V)$  is said to be compatible with the Heegaard diagram  $(\Sigma, \alpha, \beta)$  if

- $\Sigma = f^{-1}(3/2)$ ,
- $\alpha_i = U_i \cap \Sigma$  and  $\beta_j = S_j \cap \Sigma$ , for all  $1 \leq i, j \leq g$ .

An intersection point  $\mathbf{x}$  determines  $g$  flow lines  $\gamma_1, \dots, \gamma_g$  connecting the points  $p_i$  to the points  $q_j$ . The basepoint  $z$  determines a flow line  $\gamma_0$  from the index 0 critical point to the index 3 critical point. Outside the union of small neighborhoods of  $\gamma_0, \dots, \gamma_g$ , the vector field  $V$  does not vanish. The absolute grading  $\text{gr}(\mathbf{x})$  is the homotopy class of an appropriate extension of this nonvanishing vector field to the union of these small neighborhoods, as we briefly explain. Figure 1(a) illustrates two transverse vertical sections of the vector field  $V$  in a small neighborhood of  $\gamma_i$ , for some  $i \geq 1$  and

Figure 1(b) illustrates a vertical section of  $V$  in a small neighborhood of  $\gamma_0$ . Now we substitute  $V$  in these neighborhoods by the vector fields illustrated on Figure 2. We note that in the neighborhood of  $\gamma_0$ , the vector field on Figure 2(b) has a circle of zeros. We modify the vector field in neighborhood of this circle so that it rotates clockwise on the  $xy$ -plane. Then we define  $\text{gr}(\mathbf{x})$  to be the homotopy class of this vector field. For more details on this construction, see [6, §2.1].

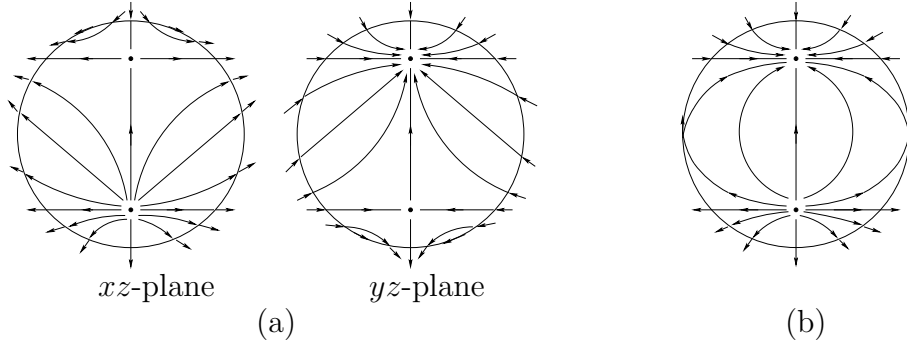


Figure 1: The vector field  $V$

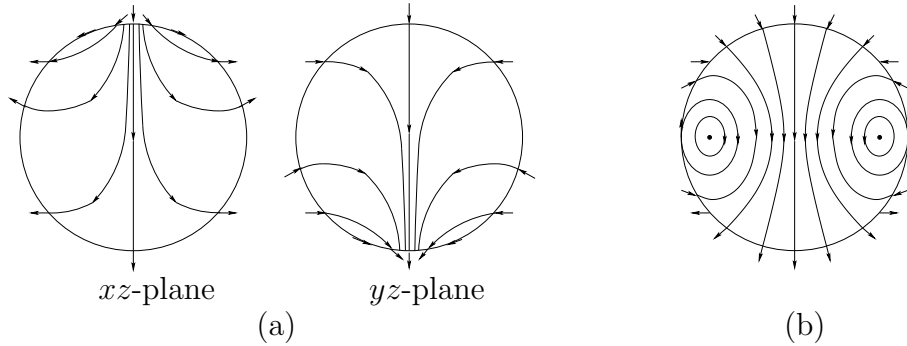


Figure 2: The modification of  $V$

Now, for an intersection point  $\mathbf{x}$  and  $n \in \mathbb{Z}$ , we define  $\text{gr}([\mathbf{x}, n]) = \text{gr}(\mathbf{x}) + 2n$ . The following was the main theorem of [6].

**Theorem 2.1.** *For a pointed Heegaard diagram  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ ,  $\text{gr}$  defines an absolute grading on any version of Heegaard Floer homology  $HF^\circ(Y)$ , satisfying the following properties:*

1. If  $\mathbf{x}$  and  $\mathbf{y}$  are in the same  $\text{Spin}^c$  structure, then  $\text{gr}(\mathbf{x}, \mathbf{y}) = \text{gr}(\mathbf{x}) - \text{gr}(\mathbf{y}) \in \mathbb{Z}/d$ , where  $d$  is the divisibility of the first Chern class of the given the  $\text{Spin}^c$  structure.
2. If  $\xi$  is a contact structure on  $Y$  and  $c(\xi)$  is the contact class in  $\widehat{HF}(-Y)$ , then  $\text{gr}(c(\xi))$  is the homotopy class of the Reeb vector field  $R_\lambda$  for any contact form  $\lambda$  for  $\xi$ .
3. The grading  $\text{gr}$  is invariant under Heegaard moves, so it does not depend on the pointed Heegaard diagram.
4. The grading  $\text{gr}$  respects the cobordism maps.

It follows from the definition that the map  $\widehat{HF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \rightarrow HF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$  in the exact triangle (2) has degree 0.

## 2.3 Heegaard Floer homology and open book decompositions

In this subsection, we will recall how to compute  $\widehat{HF}(-Y)$  from the page of an open book decomposition and we will show how to adapt the absolute grading to it. An *open book* is a pair  $(S, \varphi)$ , where  $S$  is a compact oriented surface with boundary and  $\varphi$  is a diffeomorphism of  $S$  which is the identity on  $\partial S$ . We will always assume that  $\partial S$  is connected. Given such a pair, one constructs a three-manifold as follows. Let  $N_{(S, \varphi)}$  be the mapping cylinder of  $\varphi$ , i.e. the quotient of  $S \times [0, 1]$  by the equivalence relation given by  $(x, 1) \sim (\varphi(x), 0)$ . We obtain a closed three-manifold by further quotienting  $N_{(S, \varphi)}$  by the equivalence relation given by  $(x, t) \sim (x, t')$  for all  $x \in \partial S$  and  $t, t' \in \mathbb{R}/\mathbb{Z}$ . For every closed oriented three-manifold  $Y$ , there exists an open book  $(S, \varphi)$  giving rise to  $Y$  by the above construction. So  $(S, \varphi)$  is called an open book decomposition of  $Y$ .

Let  $(S, \varphi)$  be an open book decomposition of  $Y$ . By [1, Lemma 2.1.1], we can assume that there exists a diffeomorphism of a neighborhood of  $\partial S$  in  $S$  to  $[-\varepsilon, 0] \times \partial S$  such that the monodromy  $\varphi$  is given by  $\varphi(y, \theta) = (y, \theta - y)$  in this neighborhood. Then  $(S, \varphi)$  gives rise to a Heegaard decomposition as follows. The Heegaard surface is  $\Sigma := S_{1/2} \cup -S_0$ , where  $S_t$  denotes  $S \times \{t\}$ . If we denote the genus of  $S$  by  $g$ , then  $\Sigma$  has genus  $2g$ . We choose a set of properly embedded arcs  $\mathbf{a} = \{a_1, \dots, a_{2g}\}$  of  $S$  such that  $S \setminus \bigcup_i a_i$  is a disk. One can then let  $\alpha_i = a_i^\dagger \cup a_i$ , where  $a_i$  is seen as an arc in  $S_0$  and  $a_i^\dagger$  is its

copy in  $S_{1/2}$ . One also lets  $\beta_i = \beta_i^\dagger \cup h(a_i)$ , where  $b_i^\dagger$  is the simplest arc in  $S_{1/2}$  which is isotopic to  $a_i^\dagger$  and extends  $h(a_i)$  to a smooth curve in  $\Sigma$ , see Figure 3(a). Hence  $(\Sigma, \alpha, \beta)$  is a Heegaard diagram for  $Y$ . So  $(\Sigma, \beta, \alpha)$  is a Heegaard diagram for  $-Y$ .

For each  $i$ , the circle  $a_i^\dagger$  intersects  $b_i^\dagger$  in  $S_{1/2}$  at three points. We label them  $y_i, y'_i, y''_i$ , as in Figure 3(a). We fix a basepoint  $z \in \partial S \subset \Sigma$ . One defines  $\widehat{CF}'(S, \mathbf{a}, \varphi(\mathbf{a}))$  to be the subcomplex of  $\widehat{CF}(\Sigma, \beta, \alpha, z)$  generated by  $2g$ -tuples of intersection points contained in  $S_0$ . One also defines  $\widehat{CF}(S, \mathbf{a}, \varphi(\mathbf{a}))$  to be the quotient  $\widehat{CF}'(S, \mathbf{a}, \varphi(\mathbf{a}))/\sim$ , where two  $2g$ -tuples of intersection points in  $S_0$  are equivalent if they differ by substituting  $y_i$  by  $y'_i$ . There is an induced differential on  $\widehat{CF}(S, \mathbf{a}, \varphi(\mathbf{a}))$ . It is shown in [1, Theorem 4.9.4] that the homology of this chain complex is isomorphic to  $\widehat{HF}(\Sigma, \beta, \alpha, z)$ . The absolute grading on the complex  $\widehat{CF}(S, \mathbf{a}, \varphi(\mathbf{a}))$  is well-defined as the following simple lemma shows.

**Lemma 2.2.** *If  $\mathbf{x}$  is a  $2g$ -tuple of intersection points in  $S_0$  containing  $y_i$  and  $\mathbf{x}' = (\mathbf{x} \setminus \{x_i\}) \cup \{y'_i\}$ , then  $\text{gr}(\mathbf{x}) = \text{gr}(\mathbf{x}')$ . Therefore  $\text{gr}$  is well-defined on the quotient  $\widehat{CF}(S, \mathbf{a}, \varphi(\mathbf{a}))$ .*

*Proof.* We observe that  $\mathbf{x}$  and  $\mathbf{x}'$  are in the same  $\text{Spin}^c$  structure and that  $\text{gr}(\mathbf{x}, \mathbf{x}') = 0$ . Therefore, by Theorem 2.1(a),

$$\text{gr}(\mathbf{x}) - \text{gr}(\mathbf{x}') = 0.$$

□

Moreover, by definition, the map  $\widehat{CF}'(S, \mathbf{a}, \varphi(\mathbf{a})) \rightarrow \widehat{CF}(\Sigma, \beta, \alpha, z)$  preserves the absolute grading. Therefore the isomorphism

$$\psi' : \widehat{HF}(\Sigma, \beta, \alpha, z) \rightarrow \widehat{HF}(S, \mathbf{a}, \varphi(\mathbf{a})) \quad (3)$$

preserves the absolute grading.

We observe that  $Y$  can also be decomposed as  $Y \cong N_{(S, \varphi)} \cup (S^1 \times D^2)$ , where  $\partial N_{(S, \varphi)} = \partial S \times (\mathbb{R}/\mathbb{Z}) \cong S^1 \times S^1$  is glued to  $S^1 \times \partial D^2 \cong S^1 \times S^1$ . This decomposition can also be obtained by considering the open book  $(\bar{S}, \bar{\varphi})$ , where  $\bar{S} \setminus S$  is an annulus and  $\bar{\varphi}$  is an extension of  $\varphi$  to  $\bar{S}$  which is the identity in a small neighborhood of  $\partial \bar{S}$ . Let  $\lambda$  be a contact form on  $Y$  which is *adapted* to  $(S, \varphi)$ , i.e. the Reeb vector field  $R_\lambda$  is a positively transverse to  $(\bar{S} \setminus \partial \bar{S}) \times \{t\}$  for all  $t$  and positively tangent to the binding  $\partial \bar{S}$ . For each



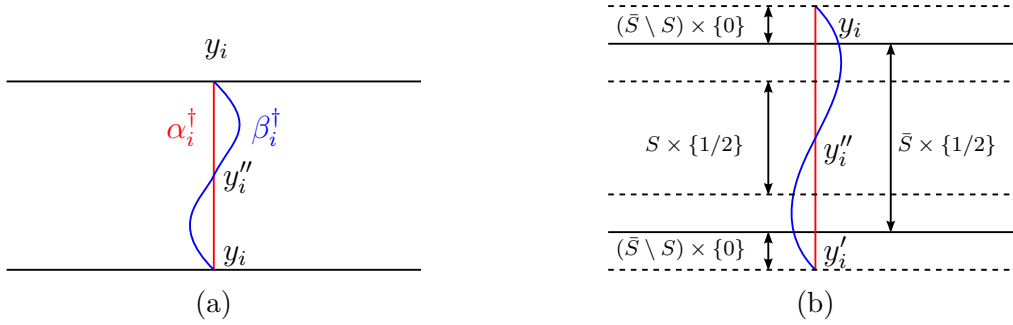


Figure 3

generator  $\mathbf{x} \in \widehat{CF}'(S, \mathbf{a}, \varphi(\mathbf{a}))$ , we will now construct a vector field  $V^\mathbf{x}$  in the homotopy class of  $\text{gr}(\mathbf{x})$  which differs from  $R_\lambda$  only in a small set.

The inclusion  $S \times \{0\} \subset \bar{S} \times \{0\}$  can be extended to a diffeomorphism  $(-\bar{S} \times \{1/2\}) \cup (\bar{S} \times \{0\}) \cong \Sigma$ , as in Figure 3(b). We will use this description of  $\Sigma$  for the rest of this section. Let  $\alpha$  and  $\beta$  be the sets of  $\alpha$  and  $\beta$  curves under this diffeomorphism. Note that  $\alpha_i \cap (S \times \{0\}) = a_i$  and that  $\beta_i \cap (S \times \{0\}) = \varphi(a_i)$ . We can also assume that  $\alpha_i \cap (S \times \{1/2\}) = a_i^\dagger$  and that  $\beta_i \cap (S \times \{1/2\}) = b_i^\dagger$  coincide outside a small neighborhood of  $\partial(S \times \{1/2\})$ , see Figure 3(b). We isotope  $z$  from  $(\partial S) \times \{0\}$  to  $(\partial S) \times \{1/2\}$  in the complement of the  $\alpha$  and  $\beta$  circles. We now choose a Morse function  $f$  and a gradient-like vector field  $V$  for  $f$  such that  $(f, V)$  is compatible with  $(\Sigma, \beta, \alpha)$  and such that the index two and one critical points are contained in  $S \times \{1/4\}$  and  $S \times \{3/4\}$ , respectively. Fix  $\varepsilon < 1/4$  and let  $M_0 = S \times ([0, \varepsilon] \cup [1 - \varepsilon, 1])$ . Since  $V$  and  $R_\lambda$  are positively transverse to  $S$  on  $M_0$ , we can assume that  $V = R_\lambda$  in  $M_0$ . For each  $i = 1, \dots, 2g$ , let  $A_i^1$  be a small neighborhood of  $\alpha_i \cap \bar{S} \times \{1/2\}$  in  $\bar{S} \times \{1/2\}$  containing  $\beta_i \cap \bar{S} \times \{1/2\}$  and let  $A_i^2 \supsetneq A_i^1$  be a small thickening of it. We can see  $A_i^1$  and  $A_i^2$  as subsets of  $\bar{S}$ . We can assume that  $(f, V)$  is chosen so that  $A_1^i \times [\varepsilon, 1 - \varepsilon]$  contains  $U_i \cup S_i$ , where  $U_i$  and  $S_i$  denote the unstable and stable manifolds of the index one and two critical points corresponding to  $\beta_i$  and  $\alpha_i$ , respectively. For  $j = 1, 2$ , we define

$$M_j = \left( \bigcup_{i=1}^{2g} A_j^i \times [0, 1] \right) \setminus M_0 \subset Y. \quad (4)$$

Let  $N(z)$  be a small neighborhood of the flow line passing through  $z$  and let  $V_0$  be the vector field obtained by modifying  $V$  in  $N(z)$  as in Figure 2(b).

We note that  $V_0$  does not vanish in  $Y \setminus M_1$ . For a generator  $\mathbf{x} \in$

$\widehat{CF}'(S, \mathbf{a}, \varphi(\mathbf{a}))$ , its grading  $\text{gr}(\mathbf{x})$  is the homotopy class of a vector field obtained by modifying  $V_0$  in  $M_0 \cup M_1$ . We will now show how to homotope  $V_0$  in the complement  $M_0 \cup M_1$  to obtain a vector field which mostly coincides with  $R_\lambda$ .

**Lemma 2.3.** *The vector field  $V_0$  is homotopic to a nonvanishing vector field  $\tilde{V}_0$  in  $Y \setminus (M_0 \cup M_1)$  relative to its boundary such that  $\tilde{V}_0$  coincides with  $R_\lambda$  in  $Y \setminus (M_0 \cup M_2)$ .*

*Proof.* The proof of this lemma is essentially the last paragraph of the proof of [6, Theorem 1.1(b)]. We will rewrite it here for the reader's sake.

Let  $B = Y \setminus (M_0 \cup M_1)$ . It follows from the definition of the arcs  $\{a_i\}$  that  $\bar{S} \setminus (\cup_i A_1^i)$  is topologically a disk with  $4g$  boundary punctures. So,  $B$  is diffeomorphic to a three-ball with  $4g$  interior punctures. We can choose this diffeomorphism so that  $B \cap (\bar{S} \times \{1/2\})$  is contained on the  $xy$ -plane in  $\mathbb{R}^3$ , as shown in Figure 4(a). Under this diffeomorphism,  $B \cap M_2$  is a union of very small punctured balls centered at the  $4g$  punctures from above. The vector field  $V_0$  is depicted in Figure 4(b). A horizontal section of  $V$  can be seen in Figure 4(c), where the green punctured disks represent  $B \cap M_2$ . We can isotope  $V_0$  so the closed orbit contained on this horizontal section coincides with the binding  $\partial \bar{S}$  outside  $M_2$ , see Figure 4(d). The vector field obtained by this isotopy can be chosen to coincide with  $R_\lambda$  in  $B \setminus M_2$ . So we let  $\tilde{V}_0$  be this vector field.  $\square$

Now we choose disjoint neighborhoods of the flow lines corresponding to each  $x_i$  which are contained in  $M_0 \cup M_1$ . We define  $V^\mathbf{x}$  to be the vector field obtained by modifying  $\tilde{V}_0$  in these neighborhoods as in §2.2. It follows from Lemma 2.3 that  $[V^\mathbf{x}] = \text{gr}(\mathbf{x}) \in \text{Vect}(Y)$ .

### 3 The grading on embedded contact homology

In this section, we will recall the definition of the ECH chain complex and its absolute grading. For more details, see [8, 9].

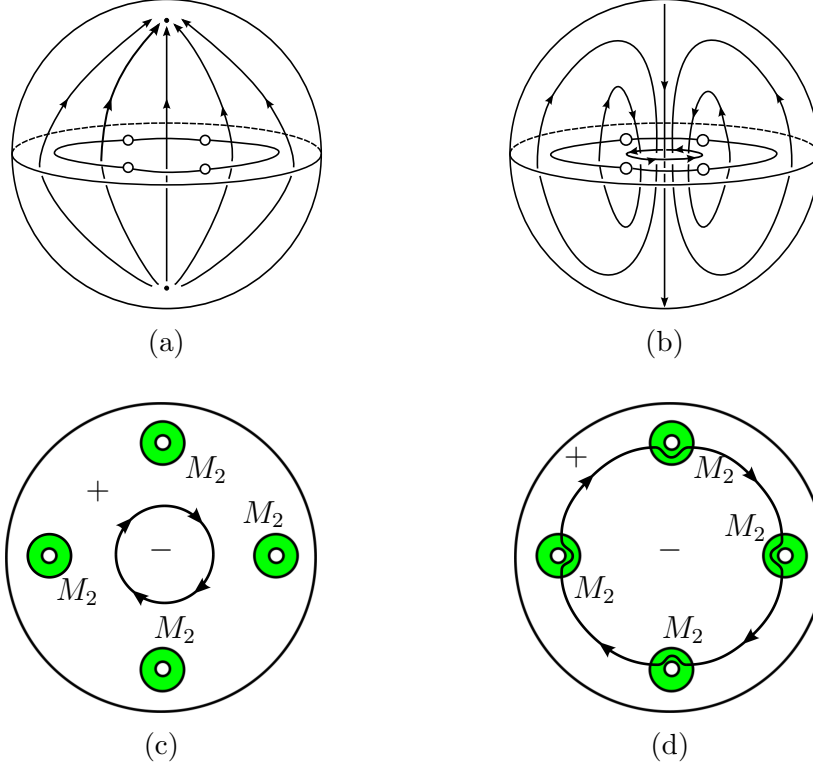


Figure 4

### 3.1 Embedded contact homology

Let  $Y$  be a closed, oriented three-manifold, let  $\lambda$  be a contact form on  $Y$  and let  $\xi = \ker(\lambda)$ . The ECH chain complex  $ECC(Y, \lambda)$  is generated by finite orbit sets  $\{(\gamma_i, m_i)\}$ , where  $\gamma_i$  are distinct embedded Reeb orbits,  $m_i$  are positive integers, and  $m_1 = 1$  whenever  $\gamma_i$  is hyperbolic. The chain complex  $ECC(Y, \lambda)$  splits by homology classes as

$$ECC(Y, \lambda) = \sum_{\Gamma \in H_1(Y)} ECC(Y, \lambda, \Gamma),$$

where  $ECC(Y, \lambda, \Gamma)$  is the subcomplex generated by the orbit sets  $\gamma = \{(\gamma_i, m_i)\}$  whose total homology class  $\sum_i m_i [\gamma_i] = \Gamma \in H_1(Y)$ . Now consider two orbit sets  $\gamma = \{(\gamma_i, m_i)\}$  and  $\sigma = \{(\sigma_j, n_j)\}$  whose total homology classes equal  $\Gamma$  and let  $Z$  be a 2-chain in  $Y \times [0, 1]$  such that

$$\partial Z = (\gamma \times \{1\}) - (\sigma \times \{0\}).$$

Let  $\tau$  denote a trivialization of  $\xi$  over all of the Reeb orbits  $\gamma_i$  and  $\sigma_j$  and let  $c_\tau(Z)$  be the relative first Chern class of  $\xi|_Z$  with respect to  $\tau$ . Let  $Q_\tau(Z)$  be the relative self-intersection number of  $Z$  with respect to  $\tau$ , as explained in [9], and let

$$CZ_\tau^I(Z) = \sum_i \sum_{k=1}^{m_i} CZ_\tau(\gamma_i^k) - \sum_j \sum_{k=1}^{n_j} CZ_\tau(\sigma_j^k),$$

where  $CZ_\tau(\rho)$  denotes the Conley-Zehnder index of the Reeb orbit  $\rho$  with respect to the trivialization  $\tau$ . In [7], Hutchings defined the *ECH index* to be

$$I(Z) = c_\tau(Z) + Q_\tau(Z) + CZ_\tau^I(Z).$$

One can show that  $I$  does not depend on  $\tau$  and only depends on the relative homology class of  $Z$ . If we change the relative homology class of  $Z$ , the ECH index  $I(Z)$  changes by a multiple of  $d$ , where  $d$  is the divisibility of  $c_1(\xi) + 2PD(\Gamma) \in H^2(Y; \mathbb{Z})$ . So,  $I(\gamma, \sigma) := I(Z)$  is well defined in  $\mathbb{Z}/d$ . Hence  $ECC(Y, \lambda, \Gamma)$  is relatively graded by  $\mathbb{Z}/d$ .

The differential  $\partial\gamma$  is defined as follows. Let  $(\mathbb{R} \times Y, d(e^s\lambda))$  be the symplectization of  $Y$ , where  $s$  denotes the  $\mathbb{R}$ -coordinate, and let  $J$  be a cylindrical almost-complex structure on  $\mathbb{R} \times Y$ . The coefficient of  $\sigma$  in  $\partial\gamma$  is the signed count of  $J$ -holomorphic curves  $C$  in  $\mathbb{R} \times Y$  that have ECH index  $I(C) = 1$ , and that converge to  $\gamma$  as  $s \rightarrow \infty$  and  $\sigma$  as  $s \rightarrow -\infty$ , modulo translation. It is shown in [10, 11] that for a generic  $J$  the differential  $\partial$  is well-defined and  $\partial^2 = 0$ . So one defines  $ECH(Y, \lambda, \Gamma, J)$  to be the homology of  $ECC(Y, \lambda, \Gamma)$  for a given  $J$ . It follows from [16] that  $ECH(Y, \lambda, \Gamma, J)$  does not depend on the choice of  $J$ , so we will omit  $J$  from the notation. We will also denote by  $ECH(Y, \lambda)$  the homology of  $ECC(Y, \lambda)$ . It follows from the definition of  $\partial$  that

$$ECH(Y, \lambda) = \sum_{\Gamma \in H_1(Y)} ECH(Y, \lambda, \Gamma).$$

Moreover, by Taubes [16], for two contact forms  $\lambda_1, \lambda_2$ , there exists an isomorphism  $ECH(Y, \lambda_1) \cong ECH(Y, \lambda_2)$ . Therefore one can write  $ECH(Y)$ . We note that there are additional structures on  $ECH(Y, \lambda)$  that do depend on the contact form, e.g. ECH capacities.

### 3.2 The absolute grading on ECH

We now recall the definition of the absolute grading on ECH from [8]. This grading takes values on the set of homotopy classes of oriented 2-plane fields

on  $Y$ . The orientation of  $Y$  induces an isomorphism from this set to the set of homotopy classes of nonvanishing vector fields  $\text{Vect}(Y)$ . Let  $\gamma = \{(\gamma_i, m_i)\}$  be an orbit set. The absolute grading  $\text{gr}(\gamma)$  is the homotopy class of the vector field obtained by modifying the Reeb vector field in disjoint neighborhoods of the Reeb orbits  $\gamma_i$ , as follows. For each  $i$ , fix a small tubular neighborhood of  $\gamma_i$  and choose a braid  $\zeta_i$  with  $m_i$  strands in that neighborhood. Let  $L$  be the union of the braids  $\zeta_i$ . A trivialization  $\tau_i$  of  $\xi$  over each  $\gamma_i$ , induces a framing  $\tau_i$  on each  $\zeta_i$ . Let  $\tau$  denote this framing on  $L$ . Now, for each component  $K$  of  $L$ , let  $N_K$  denote a small neighborhood of  $K$  in  $Y$ . We can choose these neighborhoods so that  $N_K$  and  $N_{K'}$  do not intersect for different components  $K$  and  $K'$ . The framing on  $K$  induces a diffeomorphism  $\phi_K : N_K \rightarrow S^1 \times D^2$  and a trivialization of  $TN_K$ , identifying  $\xi = \{0\} \oplus \mathbb{R}^2$  and  $R = (1, 0, 0)$ . Using the previous identifications, one can define a vector field  $P$  on  $N_K$  as

$$P : S^1 \times D^2 \rightarrow \mathbb{R} \oplus \mathbb{R}^2$$

$$(t, re^{i\theta}) \mapsto (-\cos(\pi r), \sin(\pi r)e^{-i\theta}).$$

One now constructs a vector field on  $Y$  by defining it to be given by  $P$  in each neighborhood  $N_K$  and to equal the Reeb vector field in the complement of the union of the neighborhoods  $N_K$ . Let  $P_\tau(L)$  be the homotopy class of this vector field. Now define

$$\text{gr}(\gamma) = P_\tau(L) - \sum_i w_{\tau_i}(\zeta_i) + CZ_\tau^I(\gamma), \quad (5)$$

Here  $w_{\tau_i}(\zeta_i)$  denotes the writhe of  $\zeta_i$  with respect to  $\tau_i$  and

$$CZ_\tau^I(\gamma) = \sum_i \sum_{k=1}^{m_i} CZ_\tau(\gamma_i^k).$$

One can check that  $\text{gr}(\gamma)$  does not depend on the choice of  $\tau$  or  $L$ . In [8], Hutchings proved the following proposition.

**Proposition 3.1.** *Let  $\gamma$  and  $\sigma$  be orbit sets with  $[\gamma] = [\sigma] \in H_1(Y)$  and let  $d$  denote the divisibility of  $c_1(\xi) + 2PD([\gamma]) \in H^2(Y; \mathbb{Z})$ . Then*

$$\text{gr}(\gamma) - \text{gr}(\sigma) = I(\gamma, \sigma) \in \mathbb{Z}/d.$$

For  $\rho \in \text{Vect}(Y)$ , we denote by  $ECH_\rho(Y, \lambda)$  the submodule of  $ECH(Y, \lambda)$  consisting of the elements of grading  $\rho$ . It follows from [5] that for two contact forms  $\lambda_1, \lambda_2$ , the isomorphism  $ECH(Y, \lambda_1) \cong ECH(Y, \lambda_2)$  restricts to an isomorphism  $ECH_\rho(Y, \lambda_1) \cong ECH_\rho(Y, \lambda_2)$  for every  $\rho \in \text{Vect}(Y)$ . So we can write  $ECH_\rho(Y)$  for  $\rho \in \text{Vect}(Y)$ .

### 3.3 The $U$ map and the hat version of ECH.

The  $U$  map is defined similarly to the differential. For an orbit set  $\gamma$ , one defines  $\langle U\gamma, \sigma \rangle$  to be the signed count of  $J$ -holomorphic curves  $C$  in  $\mathbb{R} \times Y$  with  $I(C) = 2$  that go through a fixed basepoint in  $\mathbb{R} \times Y$ . The  $U$  map is a degree  $-2$  chain map  $U : ECC(Y, \lambda) \rightarrow ECC(Y, \lambda)$ . The chain complex  $\widehat{ECC}(Y, \lambda, J)$  is defined to be the mapping cone of  $U$ . The homology of  $\widehat{ECC}(Y, \lambda, J)$  is denoted by  $\widehat{ECH}(Y, \lambda, J)$ . Again, it follows from [16] that the  $U$  map in homology does not depend on  $J$  or  $\lambda$  so one can write  $\widehat{ECH}(Y)$ . We obtain an exact triangle, as follows.

$$\begin{array}{ccc} ECH(Y) & \xrightarrow{U} & ECH(Y) \\ & \nwarrow \quad \nearrow & \\ & \widehat{ECH}(Y) & \end{array} \quad (6)$$

We define the absolute grading on  $\widehat{ECC}(Y, \lambda, J)$  so that the map  $\widehat{ECC}(Y, \lambda) \rightarrow ECC(Y, \lambda)$  has degree 0. Hence for  $\rho \in \text{Vect}(Y)$ , we can write  $\widehat{ECH}_\rho(Y)$ . We note that the map  $ECH(Y, \lambda) \rightarrow \widehat{ECH}(Y, \lambda)$  has degree 1.

### 3.4 Cobordism maps in ECH

In this subsection, we will show that the cobordisms maps in ECH defined by Hutchings and Taubes in [12] preserve the absolute grading. This fact will be used in the next subsection.

Let  $\lambda$  be a contact form on  $Y$ . The symplectic action of an orbit set  $\gamma = \{(m_i, \gamma_i)\}$  is defined to be  $\mathcal{A}_\lambda(\gamma) := \sum_i m_i \int_{\gamma_i} \lambda$ . For  $L > 0$ , the filtered ECH chain complex  $ECC^L(Y, \lambda)$  is defined to be the subcomplex of  $ECC(Y, \lambda)$  generated by all orbit sets  $\gamma$  with  $\mathcal{A}_\lambda(\gamma) < L$ . Since the differential decreases the action, the subgroup  $ECC^L(Y, \lambda)$  is indeed a subcomplex. Its homology is denoted by  $ECH^L(Y, \lambda)$  and it is independent of the almost-complex structure by [12, Theorem 1.3(a)].

For  $i = 1, 2$ , let  $(Y_i, \lambda_i)$  be a 3-manifold with contact form  $\lambda_i$ . An exact symplectic cobordism from  $(Y_1, \lambda_1)$  to  $(Y_2, \lambda_2)$  is a pair  $(W, d\lambda)$ , where  $W$  is a compact 4-manifold,  $d\lambda$  is a symplectic form,  $\partial W = Y_1 \cup (-Y_2)$  and  $\lambda|_{Y_i} = \lambda_i$  for  $i = 1, 2$ . According to [12, Theorem 1.9], such cobordisms induce maps

$$\Phi^L(X, \lambda) : ECH^L(Y_1, \lambda_1) \rightarrow ECH^L(Y_2, \lambda_2).$$

The maps  $\Phi^L$  are constructed by taking the composition of the corresponding map in Seiberg-Witten Floer homology and the isomorphism from ECH to Seiberg-Witten Floer homology.

**Lemma 3.2.** *Let  $(Y \times [0, 1], d\lambda)$  be an exact cobordism from  $(Y, \lambda_1)$  to  $(Y, \lambda_0)$ . Then, for every  $L > 0$ , the map  $\Phi^L(Y \times [0, 1], \lambda)$  preserves the absolute grading, i.e.  $\Phi^L(Y \times [0, 1], \lambda)$  maps  $ECH_\rho^L(Y_1, \lambda_1)$  to  $ECH_\rho^L(Y_0, \lambda_0)$  for every  $\rho \in \text{Vect}(Y)$ .*

*Proof.* The maps  $\Phi^L(Y \times [0, 1], \lambda)$  are defined as a composition of maps

$$ECH^L(Y_1, \lambda_1) \rightarrow \widehat{HM}_L(Y, \lambda_1) \rightarrow \widehat{HM}_L(Y, \lambda_0) \rightarrow ECH^L(Y, \lambda_0). \quad (7)$$

Here  $\widehat{HM}_L(Y, \lambda_1)$  and  $\widehat{HM}_L(Y, \lambda_0)$  are appropriate filtered Seiberg-Witten Floer cohomology groups, as explained in [12]. The second map in (7) is a filtered version of the cobordism maps defined in [13, §25]. Now it follows from the definition of these maps that if an element of  $\widehat{HM}_L(Y, \lambda_1)$  has grading  $\rho_1 = [v_1] \in \text{Vect}(Y)$ , then its image in  $\widehat{HM}_L(Y, \lambda_0)$  is the sum of elements of (possibly different) gradings  $\rho_0 = [v_0]$  such that for each such  $\rho_0$  there exists an almost-complex structure  $J$  on  $Y \times [0, 1]$  satisfying

$$v_i^\perp = T(Y \times \{i\}) \cap J(T(Y \times \{i\})), \quad i = 0, 1.$$

Now by considering  $\xi_t = T(Y \times \{t\}) \cap J(T(Y \times \{t\}))$ , we conclude that  $v_1^\perp$  and  $v_0^\perp$  are homotopic. Hence  $\rho_0 = \rho_1$ . So the second map in (7) preserves the absolute grading.

Now, the first and third maps in (7) preserve the grading by [5]. Therefore  $\Phi^L(Y \times [0, 1], \lambda)$  preserves the grading.  $\square$

### 3.5 ECH and open book decompositions

We now recall from [4] how to compute  $\widehat{ECH}(Y, \lambda)$  and  $ECH(Y, \lambda)$  from an open book decomposition.

We fix an open book decomposition  $(S, \varphi)$  of  $Y$  and write  $Y = N_{(S, \varphi)} \cup (S^1 \times D^2)$ , as in §2.3. Let  $N = N_{(S, \varphi)}$  and let  $t$  denote the  $[0, 1]$ -coordinate in  $N$  as in §2.3. We again assume that  $\varphi$  satisfies  $\varphi(y, \theta) = (y, \theta - y)$  in a neighborhood of  $\partial S$ . Let  $\lambda$  be a contact form on  $N$  such that  $R_\lambda$  is parallel to  $\partial/\partial t$  in  $N$ . Hence the torus  $\partial N$  is foliated by Reeb orbits. Up to a small isotopy of  $\varphi$ , we can assume that all the Reeb orbits in the interior of

$N$  are nondegenerate and that  $\partial N$  is a negative Morse-Bott torus. After a Morse-Bott perturbation, we obtain a pair of Reeb orbits  $\{e, h\}$  on  $\partial N$ . Let  $ECC^b(N, \lambda)$  be the chain complex generated by orbit sets constructed from Reeb orbits in the interior of  $N$  and  $\{e\}$  and let  $ECC(N, \lambda)$  be the chain complex generated by orbit sets constructed from Reeb orbits in the interior of  $N$  and  $\{e, h\}$ . The differential in both cases counts Morse-Bott buildings of ECH index 1. Let  $ECH^b(N, \lambda)$  and  $ECH(N, \lambda)$  denote the homology of these chain complexes. Now let  $ECH(N, \partial N, \lambda)$  and  $\widehat{ECH}(N, \partial N, \lambda)$  denote the quotients of  $ECH^b(N, \lambda)$  and  $ECH(N, \lambda)$  by the equivalence relations generated by  $[\gamma] \simeq [e\gamma]$ , respectively.

In [4], Colin, Ghiggini and Honda proved that  $ECH(N, \partial N, \lambda)$  and  $\widehat{ECH}(N, \partial N, \lambda)$  are well-defined and they constructed isomorphisms

$$\Psi_1 : ECH(N, \partial N, \lambda) \rightarrow ECH(Y), \quad (8)$$

$$\widehat{\Psi}_1 : \widehat{ECH}(N, \partial N, \lambda) \rightarrow \widehat{ECH}(Y). \quad (9)$$

We will now show that  $ECH(N, \partial N, \lambda)$  and  $\widehat{ECH}(N, \partial N, \lambda)$  have well-defined gradings and that (8) and (9) preserve the gradings.

We start by recalling the construction of the isomorphism  $\Psi_1$ . We write  $S^1 \times D^2 \cong V \cup (T^2 \times [0, 1])$  where  $V$  is a tubular neighborhood of the binding  $S^1 \times \{0\}$ , which is again a solid torus. Let  $\lambda_V$  be a contact form on  $V$  which is nondegenerate in the interior of  $V$  such that the Reeb vector field of  $\lambda_V$  is positively transverse to the interior of the pages and positively tangent to the binding and such that  $\partial V$  is a *positive* Morse-Bott torus. The precise construction of  $\lambda_V$  will not be necessary here and we refer the reader to [4, §8.1]. We denote by  $e'$  and  $h'$  the elliptic and hyperbolic orbits obtained after a Morse-Bott perturbation. Let  $\{L_k\}$  be an increasing sequence such that  $\lim_{k \rightarrow \infty} L_k = \infty$ . Following [4, §9.3], we can choose a family of contact forms  $\{\lambda_k\}$  on  $Y$  which equal  $\lambda$  in a neighborhood of  $N$  and a positive multiple of  $\lambda_V$  in a neighborhood of  $V$  such that  $\lambda_k$  is a Morse-Bott contact form and all Reeb orbits in  $T^2 \times [0, 1]$  have action larger than  $L_k$ . So as in [4, §9.2], we can perturb  $\{\lambda_k\}$  to a sequence of contact forms  $\{\lambda'_k\}$  satisfying, in particular, the following conditions:

- $\lambda'_k$  coincides with  $\lambda_k$  outside neighborhoods of the Morse-Bott tori.
- The Reeb orbits of  $\lambda_k$  of action less than  $L_k$  are nondegenerate and are either the Reeb orbits of  $\lambda$  and  $\lambda_V$  in the interior of  $N$  and  $V$ , respectively, or one of the orbits  $e, h, e'$  or  $h'$ .



Hence  $ECC^{L_k}(Y, \lambda'_k)$  is generated by elements of the form  $\gamma_V \cdot \gamma_N$ , where  $\gamma_V$  is an orbit set constructed from Reeb orbits in the interior of  $V$  and  $\{e', h'\}$ , and  $\gamma_N$  is a generator of  $ECC(N, \lambda)$ . For  $L > 0$ , let  $ECC_{\leq k}^{b, L}(N, \lambda)$  be the subcomplex of  $ECC^b(N, \lambda)$  generated by orbit sets  $\gamma$  with action  $\int_\gamma \lambda < L$  and whose total homology class intersects a page up to  $k$  times. Following [4, §9.7], we can define another increasing sequence  $\{L'_k\}$  with  $\lim_{k \rightarrow \infty} L'_k = \infty$  such that the maps  $\sigma_k$  below are well-defined.

$$\begin{aligned} \sigma_k : ECC_{\leq k}^{b, L'_k}(N, \lambda) &\rightarrow ECC^{L_k}(Y, \lambda'_k) \\ \gamma &\mapsto \sum_{i=0}^{\infty} (e')^i \cdot (\partial'_N)^i \gamma. \end{aligned}$$

Here  $\partial'_N \gamma$  is defined by the equation  $\partial_N \gamma = \partial_N^b \gamma + h \partial'_N \gamma$ , where  $\partial_N$  and  $\partial_N^b$  are the differentials in  $ECC(N, \lambda)$  and  $ECC^b(N, \lambda)$ , respectively. It follows from [4, Lemma 9.7.2] that the maps  $\sigma_k$  are chain maps so they induce maps  $\sigma_k : ECH_{\leq k}^{b, L'_k}(N, \lambda) \rightarrow ECH^{L_k}(Y, \lambda'_k)$ .

Following [4, Cor. 3.2.3], there are chain maps  $\Phi_k : ECC^{L_k}(Y, \lambda'_k) \rightarrow ECC^{L_{k+1}}(Y, \lambda'_{k+1})$  which are given by cobordism maps as in §3.4. So we obtain a directed system

$$\begin{array}{ccc} ECC_{\leq k}^{b, L'_k}(N, \lambda) & \xrightarrow{\sigma_k} & ECC^{L_k}(Y, \lambda'_k) \\ \downarrow \iota_k & & \downarrow \Phi_k \\ ECC_{\leq k+1}^{b, L'_{k+1}}(N, \lambda) & \xrightarrow{\sigma_k} & ECC^{L_{k+1}}(Y, \lambda'_{k+1}) \end{array} \quad (10)$$

where  $\iota_k$  denotes the inclusion. The maps  $\Phi_k$  induce maps in homology with respect to which one can take the direct limit  $\lim_{k \rightarrow \infty} ECH^{L_k}(Y, \lambda'_k)$ . There is also a nondegenerate contact form  $\lambda_0$  and cobordism maps  $ECH^{L_k}(Y, \lambda_0) \rightarrow ECH^{L_k}(Y, \lambda'_k)$ . It is shown in [4, Cor. 3.2.3] that the direct limit of these maps is an isomorphism

$$ECH(Y, \lambda_0) \cong \lim_{k \rightarrow \infty} ECH^{L_k}(Y, \lambda'_k). \quad (11)$$

Now we note that  $ECH^b(N, \lambda) = \lim_{k \rightarrow \infty} ECH_{\leq k}^{b, L'_k}(N, \lambda)$ . Therefore the maps  $\sigma_k$  give rise to a map

$$\bar{\sigma} : ECH^b(N, \lambda) \rightarrow \lim_{k \rightarrow \infty} ECH^{L_k}(Y, \lambda'_k) \cong ECH(Y, \lambda_0).$$

The calculations in [4, §9.7] imply that the image of the map  $ECH^b(N, \lambda) \rightarrow ECH^b(N, \lambda)$  given by  $[\gamma] \mapsto [\gamma] - [e\gamma]$  is contained in the kernel of  $\bar{\sigma}$ . Hence we obtain a map

$$\Psi_1 : ECH(N, \partial N, \lambda) \rightarrow ECH(Y).$$

It is shown in [4, Theorem 9.8.3] that  $\Psi_1$  is an isomorphism.

We will now prove a lemma that will be useful to show that the absolute grading is well-defined in  $ECH(N, \partial N, \lambda)$  and that  $\Psi_1$  preserves the grading.

**Lemma 3.3.** *Let  $\gamma$  be an orbit set obtained from the Reeb orbits of  $\lambda$  in the interior of  $N$ , respectively, and the orbits  $e, h, e'$  or  $h'$ . Then  $\text{gr}(\gamma) \in \text{Vect}(Y)$  is well-defined. Moreover,*

$$\begin{aligned} \text{gr}(e\gamma) &= \text{gr}(\gamma), & \text{gr}(h\gamma) &= \text{gr}(\gamma) + 1, \\ \text{gr}(e'\gamma) &= \text{gr}(\gamma) + 2, & \text{gr}(h'\gamma) &= \text{gr}(\gamma) + 1. \end{aligned} \tag{12}$$

*Proof.* To see that  $\gamma$  has a well-defined grading, first note that there exists  $k_0$  such that  $\gamma \in ECC^{L_k}(Y, \lambda'_k)$  for every  $k \geq k_0$ . So we define  $\text{gr}(\gamma)$  using the contact form  $\lambda'_k$  for some  $k \geq k_0$ . It follows from Lemma 3.2 that the maps  $\Phi_k$  preserve the grading. So  $\text{gr}(\gamma) \in \text{Vect}(Y)$  is well-defined.

To prove (12), we can restrict to the case when  $\gamma$  does not contain  $e, h, e'$  or  $h'$ . The general case is a straightforward consequence of this case. Let  $\tau$  be a trivialization of  $\xi$  over  $\gamma$  and let  $L$  be a link as in §3.2 so that  $\text{gr}(\gamma) = P_\tau(L) - w_\tau(L) + CZ_\tau^I(\gamma)$ , where  $w_\tau(L)$  denotes the sum of the writhes of all components of  $L$ . Let  $x \in \{e, h, e', h'\}$ . We can choose a trivialization  $\eta$  of  $\xi|_x$  that is trivial with respect to the Morse-Bott torus containing  $x$ . Let  $\zeta$  be a knot obtained by pushing  $x$  in a direction which is transverse to the Morse-Bott torus containing  $x$  such that  $\zeta$  is in the interior of  $Y \setminus N$ . Then  $w_\eta(\zeta) = 0$ . Now let  $D$  be a disk in  $Y \setminus N$  bounding  $x$ . It follows from [8, Lemma 3.4(d)] that

$$P_{(\tau, \eta)}(L \cup \zeta) - P_\tau(L) = c_1(\xi|_D, \eta) = 1.$$

Moreover,

$$\begin{aligned} CZ_\eta(x) &= -1, \text{ if } x = e, \\ CZ_\eta(x) &= 0, \text{ if } x = h, h', \\ CZ_\eta(x) &= 1, \text{ if } x = e'. \end{aligned}$$

Therefore it follows from (5) that (12) holds.  $\square$

**Proposition 3.4.** *The module  $ECH(N, \partial N, \lambda)$  has a well-defined absolute grading and the isomorphism  $\Psi_1 : ECH(N, \partial N, \lambda) \rightarrow ECH(Y)$  preserves the grading.*

*Proof.* It follows from Lemma 3.3 that the grading on  $ECH^b(N, \lambda)$  is well-defined. We recall that  $ECH(N, \partial N, \lambda)$  is the quotient of  $ECH^b(N, \lambda)$  by the equivalence relation given by  $[\gamma] \sim [e\gamma]$ . By Lemma 3.3 that  $\text{gr}(\gamma) = \text{gr}(e\gamma)$ . So the grading on  $ECH(N, \partial N, \lambda)$  is well-defined.

Let  $\gamma$  be an orbit set in  $ECC_{\leq k}^{b, L'_k}(N, \lambda)$  for some  $k$ . Since  $\partial_N$  decreases the grading by 1, it follows that  $\text{gr}(h\partial'_N \gamma) = \text{gr}(\gamma) - 1$ . Now, by Lemma 3.3,  $\text{gr}(\partial'_N \gamma) = \text{gr}(\gamma) - 2$ . Hence for all  $0 \geq i \geq k$ ,

$$\text{gr}((e')^i \cdot (\partial'_N)^i \gamma) = \text{gr}(\gamma) - 2i + 2i = \text{gr}(\gamma).$$

So  $\sigma_k$  preserves the grading. Now, it is tautological that the inclusion  $\iota_k$  in (10) preserves the grading. Moreover, by Lemma 3.2, the maps  $\Phi_k$  and the isomorphism (11) preserve the grading. Hence after passing to homology and taking the direct limit we conclude that  $\bar{\sigma}$ , and hence  $\Psi_1$ , preserve the grading.  $\square$

It also follows from Lemma 3.3 that the gradings on  $ECH(N, \lambda)$  and  $\widehat{ECH}(N, \partial N, \lambda)$  are well-defined. We now define two chain maps as follows.

$$\begin{array}{ccc} \iota : ECC^b(N, \lambda) & \longrightarrow & ECC(N, \lambda) \\ \gamma & \longmapsto & h\gamma \end{array} \quad \begin{array}{ccc} \pi : ECC(N, \lambda) & \longrightarrow & ECC^b(N, \lambda) \\ \gamma_1 + h\gamma_2 & \longmapsto & \gamma_1 \end{array} \quad (13)$$

Here  $\gamma_1$  and  $\gamma_2$  do not contain  $h$ . These maps descend to homology and to the quotients  $\widehat{ECH}(N, \partial N, \lambda)$  and  $ECH(N, \partial N, \lambda)$ . It follows from [4, §9.9] that these maps fit into an exact triangle

$$\begin{array}{ccc} ECH(N, \partial N, \lambda) & \xrightarrow{\quad\quad\quad} & ECH(N, \partial N, \lambda) \\ & \nwarrow \pi_* \quad \nearrow \iota_* & \\ & \widehat{ECH}(N, \partial N, \lambda) & \end{array} \quad (14)$$

where the map  $ECH(N, \partial N, \lambda) \rightarrow ECH(N, \partial N, \lambda)$  is a version of the  $U$  map. Moreover there exists an isomorphism  $\widehat{\Psi}_1 : ECH(N, \partial N, \lambda) \rightarrow \widehat{ECH}(Y)$  such that  $\Psi_1$  and  $\widehat{\Psi}_1$  give an isomorphism from (14) to (6). It follows from

(13) that  $\iota_*$  increases the grading by 1 and that  $\pi_*$  preserves the grading. Hence we obtain the following long exact sequences.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & ECH_{\rho-1}(N, \partial N) & \longrightarrow & \widehat{ECH}_{\rho}(N, \partial N) & \longrightarrow & ECH_{\rho}(N, \partial N) \longrightarrow \cdots \\
& & \downarrow \Psi_1 & & \downarrow \widehat{\Psi}_1 & & \downarrow \Psi_1 \\
\cdots & \xrightarrow{U} & ECH_{\rho-1}(Y) & \longrightarrow & \widehat{ECH}_{\rho}(Y) & \longrightarrow & ECH_{\rho}(Y) \xrightarrow{U} \cdots
\end{array} \tag{15}$$

Here  $\rho \in \text{Vect}(Y)$  and we dropped the dependence on  $\lambda$ . In fact, because of the isomorphisms  $\Psi_1$  and  $\widehat{\Psi}_1$ , the modules on the upper row of (15) do not depend on  $\lambda$ . It follows from (15) that  $\widehat{\Psi}_1$  preserves the grading.

We end this section with a discussion of  $ECC_{2g}(N, \lambda)$  and the grading on it. The chain complex  $ECC_{2g}(N, \lambda)$  is a subcomplex of  $ECC(N, \lambda)$  generated by orbits sets which intersect  $S$  exactly  $2g$  times where  $g$  is the genus of  $S$ . This inclusion induces an absolute grading on  $ECC_{2g}(N, \lambda)$  and a map in homology. By composing this map with the quotient map  $ECH(N, \lambda) \rightarrow \widehat{ECH}(N, \partial N, \lambda)$  we obtain a map  $\widehat{\Psi}_2 : ECH_{2g}(N, \lambda) \rightarrow \widehat{ECH}(N, \partial N, \lambda)$  which preserves the grading. It is shown in [2] that  $\widehat{\Psi}_2$  is an isomorphism. Let  $\psi = \widehat{\Psi}_1 \circ \widehat{\Psi}_2$ . Therefore  $\psi : ECH_{2g}(N, \lambda) \rightarrow \widehat{ECH}(Y)$  is an isomorphism and it preserves the grading.

## 4 The main theorem

In this section, we will prove that the absolute grading is preserved under the isomorphism from Heegaard Floer homology to ECH defined by Colin-Ghiggini-Honda in [1, 2, 3].

### 4.1 The construction of $\widetilde{\Phi}$

We now recall the construction of the map  $\widetilde{\Phi}$  on the chain level

$$\widetilde{\Phi} : \widehat{CF}(S, \mathbf{a}, \varphi(\mathbf{a})) \rightarrow ECC_{2g}(N, \lambda).$$

This map is defined by counting rigid holomorphic curves with an ECH-type index equal to 0. We now review the relevant moduli spaces and this ECH-type index.

Throughout this section we fix an open book  $(S, \varphi)$  for  $Y$  satisfying the conditions given in §2.3 and we let  $N := N_{(S, \varphi)}$  be the mapping torus of  $\varphi$ . We denote by  $g$  the genus of  $S$  and we let  $\lambda$  be a contact form on  $Y$  which is adapted to  $(S, \varphi)$ . In order to prove that  $\tilde{\Phi}$  is an isomorphism, it is necessary to make a more specific choice of  $\lambda$  as it is done in [1, §3], but this particular choice does not affect the absolute grading.

Let  $\pi : \mathbb{R} \times N \rightarrow \mathbb{R} \times S^1$  be the map  $(s, x, t) \mapsto (s, t)$  and let  $B := (\mathbb{R} \times S^1) \setminus B^c$ , where  $B^c = (0, \infty) \times (1/2, 1)$ . We also round the corners of  $B$ . Now define  $W = \pi^{-1}(B)$  and  $\Omega = ds \wedge dt + \omega$ , where  $\omega$  is a certain area form on  $S$ . Then  $(W, \Omega)$  is a symplectic manifold with boundary. It has a positive end, which is diffeomorphic to  $S \times [0, 1/2]$  and a negative end, which is diffeomorphic to  $N$ . The map  $\pi$  restricts to a symplectic fibration  $\pi_B : (W, \Omega) \rightarrow (B, ds \wedge dt)$  which admits a symplectic connection whose horizontal space is spanned by  $\{\partial/\partial s, \partial/\partial t\}$ . Now if we take a copy of  $\mathbf{a}$  on the fiber  $\pi_B^{-1}(1, 1/2)$  and take its symplectic parallel transport along  $\partial B$ , we obtain a Lagrangian submanifold of  $(W, \Omega)$ , which is denoted by  $L_{\mathbf{a}}$ . For each  $a_i \subset \mathbf{a}$  we denote by  $L_{a_i}$  the corresponding component of  $L_{\mathbf{a}}$ .

We will consider  $J$ -holomorphic maps  $u : (\dot{F}, j) \rightarrow (W, J)$  where  $(\dot{F}, j)$  is a Riemann surface with boundary and punctures, both in the interior and on the boundary. A puncture  $p$  is said to be positive or negative if the  $s$ -coordinate of  $u(x)$  converges to  $\infty$  or  $-\infty$ , respectively, as  $x \rightarrow p$ . Now to each generator  $\mathbf{x}$  of  $\widehat{CF}(S, \mathbf{a}, \varphi(\mathbf{a}))$  we can associate a subset of  $S \times [0, 1/2]$  given by the union of  $x_i \times [0, 1/2]$ , for all  $x_i \in \mathbf{x}$ . We will still denote the union of these chords by  $\mathbf{x}$ . Given  $\mathbf{x}$ , an orbit set  $\gamma = \{(\gamma_i, m_i)\}$  in  $ECC_{2g}(N, \lambda)$  and an admissible almost-complex structure  $J$ , one defines  $\mathcal{M}_J(\mathbf{x}, \gamma)$  to be the moduli space of  $J$ -holomorphic maps  $u : (\dot{F}, j) \rightarrow (W, J)$  satisfying the following conditions:

- (a)  $u(\partial \dot{F}) \subset L_{\mathbf{a}}$  and each component of  $\partial \dot{F}$  is mapped to a different  $L_{a_i}$ .
- (b) The boundary punctures are positive and the interior punctures are negative.
- (c) At each boundary puncture,  $u$  converges to a different chord  $x_i \times [0, 1/2]$  and every chord  $x_i \times [0, 1/2]$  is such an end of  $u$ .
- (d) At an interior puncture,  $u$  converges to an orbit  $\gamma_i$  with some multiplicity. For each  $i$ , the total multiplicity of all ends converging to  $\gamma_i$  is  $m_i$ .

(e) The energy of  $u$  is bounded.

Let  $\overline{W}$  denote the compactification of  $W \subset \mathbb{R} \times N$  obtained by compactifying  $\mathbb{R}$  to  $\mathbb{R} \cup \{-\infty, \infty\}$ . A continuous map  $u : \dot{F} \rightarrow W$  satifying (a)–(d) above can be compatified to a map  $\bar{u} : \overline{F} \rightarrow \overline{W}$  mapping  $\partial \overline{F}$  to  $L_{\mathbf{x}, \gamma} := L_{\mathbf{a}} \cup (\mathbf{x} \times \{\infty\}) \cup (\gamma \times \{-\infty\})$ . Two such maps  $u, v$  are said to be homologous if the images of  $\bar{u}$  and  $\bar{v}$  are homologous in  $H_2(\overline{W}, L_{\mathbf{x}, \gamma})$ . Let  $H_2(W, \mathbf{x}, \gamma)$  denote the set of homology classes of such maps  $u : \dot{F} \rightarrow W$ .

For a homology class  $A \in H_2(W, \mathbf{x}, \gamma)$ , one defines its ECH-index  $I(A)$  as follows. Let  $u : \dot{F} \rightarrow W$  be a continuous map satifying (a)–(d) above such that  $[u] = A$  and let  $\bar{u} : \overline{F} \rightarrow \overline{W}$  be its compactification. Now note that one can view  $TS$  as a sub-bundle of  $T\overline{W}$ . We choose an orientation of the arcs  $a_i$ , which gives rise to a nonvanishing vector field along each  $a_i$ . This vector field induces a trivialization  $\tau$  of  $TS$  along  $L_{\mathbf{a}} \subset \overline{W}$ . We extend this trivialization arbitrarily along  $\{\infty\} \times \mathbf{x} \times [0, 1/2]$  and along  $\{-\infty\} \times \gamma$ . Let  $c_\tau(A)$  denote the first Chern class of  $\bar{u}^*TS$  relative to  $\tau$ . Now let  $C_1$  and  $C_2$  be distinct embedded surfaces in  $\overline{W}$  given by pushing  $\bar{u}(\overline{F})$  off along vectors field which are transverse to it and trivial with respect to  $\tau$  along the boundary. For more details see [1, §4]. Then  $Q_\tau(A)$  is defined to be the signed count of intersections of  $C_1$  and  $C_2$ . Now let  $\mathcal{L}_0$  be a real, rank one subbundle of  $TS$  along  $\mathbf{x} \times [0, 1/2]$  defined as follows. At  $\mathbf{x} \times \{0\}$ , let  $\mathcal{L}_0 = T\varphi(\mathbf{a})$  and at  $\mathbf{x} \times \{1/2\}$ , let  $\mathcal{L}_0 = T\mathbf{a}$  in  $TS$ . Then  $\mathcal{L}_0$  is defined by rotating counterclockwise by the minimum possible amount as we travel along  $\mathbf{x} \times [0, 1/2]$ . One defines  $\mu_\tau(\mathbf{x})$  to be the sum of the Maslov indices of  $\mathcal{L}_0$  along each  $x_i \times [0, 1/2]$  with respect to  $\tau$ . The ECH-index is defined as

$$I(A) = c_\tau(A) + Q_\tau(A) + \mu_\tau(\mathbf{x}) - CZ_\tau^I(\gamma) - 2g.$$

Now  $\tilde{\Phi}(\mathbf{x})$  is defined as follows. The coefficient of an orbit set  $\gamma$  in  $\tilde{\Phi}(\mathbf{x})$  is the (signed) count of maps  $u$  in  $\mathcal{M}_J(\mathbf{x}, \gamma)$  with  $I([u]) = 0$ . As explained in [1], for a generic  $J$  this count is finite and all the maps that are counted are embeddings. Showing that the map  $\tilde{\Phi}$  induces an isomorphism on homology is far from trivial and it is done in [1, 2]. We remark that [1, 2], they only construct  $\tilde{\Phi}$  over  $\mathbb{Z}/2$  coefficients. Our construction and, in particular Proposition 4.1, do not depend on an eventual choice of signs.

## 4.2 The proof of the main theorem

We now prove a proposition, which is the main result of this paper.

**Proposition 4.1.** *Let  $A \in H_2(W, \mathbf{x}, \gamma)$ , where  $\mathbf{x}$  and  $\gamma$  are generators of  $\widehat{CF}(S, \mathbf{a}, \varphi(\mathbf{a}))$  and  $ECC_{2g}(N, \lambda)$ , respectively. Then*

$$\text{gr}(\mathbf{x}) - \text{gr}(\gamma) = I(A). \quad (16)$$

We first recall a relative version of the Pontryagin-Thom construction. Let  $v$  and  $w$  be nonvanishing vector fields on a closed and oriented three-manifold  $Y$ . Assume that  $v$  and  $w$  coincide in  $Y \setminus U$ , where  $U$  is an open set in  $Y$ . Let  $\tau$  be a trivialization of  $TY|_U$  and let  $p \in S^2$  be a regular value of both  $v$  and  $w$  seen as maps  $U \rightarrow S^2$ . The one-manifolds  $L_v := v^{-1}(p)$  and  $L_w := w^{-1}(p)$  inherit framings by considering the isomorphisms of the normal bundles with  $T_p S^2$  given by  $v_*$  and  $w_*$  along  $L_v$  and  $L_w$ , respectively. Now if  $L_v$  and  $L_w$  are contained in the interior of  $U$  and are homologous in  $U$ , there is a link cobordism  $C \subset U \times [0, 1]$  from  $L_v$  to  $L_w$ . That is,  $C$  is a surface such that  $\partial C = (L_v \times \{1\}) \cup (-L_w \times \{0\})$ . The framings on  $L_v$  and  $L_w$  induce a framing on  $C$  along  $\partial C$  which we denote by  $\nu$ .

**Lemma 4.2.** *Let  $v$  and  $w$  be nonvanishing vector fields and  $L_v$  and  $L_w$  the links as above. Let  $C$  be an immersed cobordism from  $L_v$  to  $L_w$  and let  $\delta(C)$  denote the number of self-intersections of  $C$ . Let  $\nu$  denote the framing on  $C$  along  $\partial C$  which is induced by the framings on  $L_v$  and  $L_w$ . Then*

$$[v] - [w] = c_1(NC, \nu) + 2\delta(C).$$

*Proof.* First assume that  $C$  is an embedded surface. We can find a framing  $\tilde{\nu}$  of  $C$  which coincides with  $\nu$  along  $L_w \times \{0\}$ . It follows from the Pontryagin-Thom construction that  $[v] - [w]$  equals the difference of the framings  $\tilde{\nu}$  and  $\nu$  along  $L_v \times \{1\}$ . But this difference is given by  $c_1(NC, \nu)$ . Now the general case follows from [6, Lemma 2.3].  $\square$

*Proof of Proposition 4.1.* We write  $\gamma = \{(\gamma_i, m_i)\}$ . Let  $u : \dot{F} \rightarrow W$  be an immersion such that  $[u] = A$  and let  $\bar{u} : \bar{F} \rightarrow \bar{W}$  denote its continuous compactification. We note that by rounding the corners of  $\bar{W}$ , we obtain a trivial cobordism from  $N$  to itself which we denote by  $X$ . Here we identify  $X \simeq N \times [0, 1]$ . Let  $L$  be the union of disjoint braids  $\zeta_i$  around  $\gamma_i$  with total multiplicity  $m_i$ . Then by moving  $\bar{u}(\partial \bar{F})$  in the direction of some transverse vector field near  $N \times \{0\}$ , we obtain an immersed cobordism  $C$  from  $\mathbf{x}' \times \{1\}$  to  $L \times \{0\}$ , where  $\mathbf{x}'$  is the union of  $\mathbf{x}$  with segments on the arcs  $a_i$ . Up to an isotopy, we can assume that  $\mathbf{x}' \times \{1\}$  is transverse to  $S \times \{t\} \times \{1\}$ . We also note that, under our identification,  $L_{\mathbf{a}} \subset (S \times [1/2, 1]) \times \{1\}$ .

Let  $M_2 \subset Y$  be the subset defined by (4). Let  $V^{\mathbf{x}}$  be the vector field defined in §2.3 and let  $V_\tau^L$  be the vector field defined in §2.2 whose homotopy class is  $P_\tau(L)$ . Let  $\tilde{N}$  be a small open neighborhood of  $N \cup M_2$ . Then, by Lemma 2.3, we can assume that  $V^{\mathbf{x}}$  and  $V_\tau^L$  coincide in  $Y \setminus \tilde{N}$ . We will now use Lemma 4.2. We first choose a nonvanishing tangent vector field along the arcs  $a_i$ . That induces a trivialization of  $TS|_{L_{\mathbf{a}}}$ . Since  $TS|_N \cong \xi|_N$ , we obtain a trivialization of  $\xi|_{L_{\mathbf{a}}}$ . We extend it arbitrarily to  $\mathbf{x} \times \{1\}$ . We can now extend this trivialization to a trivialization  $\tau$  of  $\xi = \ker(\lambda)$  on  $\tilde{N} \times \{1\} \cong \tilde{N}$ . So  $(\tau, R_\lambda)$  is a trivialization of  $T\tilde{N}$  where the Reeb vector field  $R_\lambda$  in  $\tilde{N}$  is mapped to  $(0, 0, 1) \in \mathbb{R}^3$ . We observe that  $(V_\tau^L)^{-1}(0, 0, -1) = L$ . The framing can be calculated by considering the preimage of a vector near  $(0, 0, -1)$ . The corresponding link gives a framing of the normal bundle  $NL \cong \xi|_L$  which coincides with  $\tau$ .

Now we compute  $(V^{\mathbf{x}})^{-1}(0, 0, -1)$ . First note that this link is contained in  $M_2$ . We observe that  $(V^{\mathbf{x}})^{-1}(0, 0, -1)$  is a link  $L_{\mathbf{x}}$  which is a slight perturbation of the union of the flow lines corresponding to the points  $x_i \in \mathbf{x}$  and the points  $y_i'' \in S \times \{1/2\}$  from Figure 3(b). The framing on  $L_{\mathbf{x}}$  is a trivialization of  $NL_{\mathbf{x}} \cong \xi|_{L_{\mathbf{x}}}$  which we denote by  $\nu$ . The link  $L_{\mathbf{x}}$  is isotopic to  $\mathbf{x}'$  so we can assume that  $C \cap (N \times \{1\}) = L_{\mathbf{x}} \times \{1\}$ . So we can see  $\nu$  as a trivialization of  $NC$  along  $L_{\mathbf{x}} \times \{1\}$ . As in Lemma 4.2, the framing on  $L \times \{0\}$  is also denoted by  $\nu$ . By Lemma 4.2,

$$\text{gr}(\mathbf{x}) - P_\tau(L) = c_1(NC, \nu) + 2\delta(C). \quad (17)$$

The trivialization  $\tau$  also gives rise to a trivialization of  $NC$  along  $L_{\mathbf{x}} \times \{1\}$  and  $L \times \{0\}$ . We claim that

$$c_1(NC, \nu) = c_1(NC, \tau) + \mu_\tau(\mathbf{x}) - 2g. \quad (18)$$

To prove the claim, we first compute the difference  $c_1(NC, \nu) - c_1(NC, \tau)$ . This difference is given by  $\nu - \tau$  as framings of  $L_{\mathbf{x}} \subset N$ , since  $\nu = \tau$  along  $L \times \{0\}$ . We will now identify  $L_{\mathbf{x}} \cong \gamma_{\mathbf{x}} \cup \gamma_{\mathbf{y}''}$ , where  $\gamma_{\mathbf{x}}$  is the union of the flow lines corresponding to  $\mathbf{x}$  and  $\gamma_{\mathbf{y}''}$  is the union of the flow lines corresponding to all  $y_i''$ . Let  $v_\tau$  be a vector field along  $L_{\mathbf{x}}$  which is trivial with respect to  $\tau$  and which coincides with the vector field tangent to each  $a_i$  as chosen above along  $\gamma_{\mathbf{y}''}$ . It follows from the isotopy  $L_{\mathbf{x}} \cong \mathbf{x}'$  and from the definition of  $\tau$  that at the index two or one critical points,  $v_\tau$  is tangent to the stable or unstable submanifolds, respectively. Moreover, the vector field  $v_\tau$  rotates a quarter of a turn counterclockwise about each component of  $\gamma_{\mathbf{x}''}$  as we go from



an index two to an index one critical point. So along each component of  $\gamma_{\mathbf{y}''}$ , the trivializations  $\nu$  and  $\tau$  differ by a half-turn clockwise. Now we compute the difference  $\nu - \tau$  along each component of  $\gamma_{\mathbf{x}}$  as we go from an index one to an index two critical point. If  $v_\tau$  rotates a quarter of a turn counterclockwise about a certain component, we again obtain a contribution of  $-1/2$  to  $\nu - \tau$ . In that case, this component will contribute by 0 to  $\mu_\tau(\mathbf{x})$ . Now say that  $v_\tau$  differs from a quarter of a turn counterclockwise rotation by  $n$  counterclockwise half-turns, for some  $n \in \mathbb{Z}$ . Then we obtain a contribution of  $-1/2 - n$  to  $\nu - \tau$  and  $-n$  to  $\mu_\tau(\mathbf{x})$ . Therefore the total difference  $\nu - \tau$  along  $L_{\mathbf{x}}$  is  $\mu_\tau(\mathbf{x}) - 2g$  and we have proven (18).

It remains to compute  $c_1(NC, \tau)$ . For that, we will use a classical construction in topology. Consider a generic section of  $NC \rightarrow C$  which is trivial with respect to  $\tau$  along  $\partial C$ . We move  $C$  in the direction of this section and we obtain a surface  $C'$  which intersects  $C$  transversely. Then

$$c_1(NC, \tau) = C \cdot C' - 2\delta(C), \quad (19)$$

where  $C \cdot C'$  denotes the signed count of intersections of  $C$  and  $C'$ . Now  $C$  and  $C'$  can be completed to surfaces in the homology class of  $A$ . But these surfaces are not necessarily  $\tau$ -trivial. In fact, the linking number of  $\partial C$  and  $\partial C'$  in  $N \times \{0\}$  is  $\sum_i w_\tau(\zeta_i)$ . Following a standard calculation in ECH, see e.g. [8, §2.7], we obtain

$$C \cdot C' = Q_\tau(A) + \sum_i w_\tau(\zeta_i). \quad (20)$$

Recall that

$$\text{gr}(\gamma) = P_\tau(L) - \sum_i w_\tau(\zeta_i) + CZ_\tau^I(\gamma). \quad (21)$$

By combining (17), (18), (19), (20) and (21), we obtain (16).  $\square$

We can now prove the main theorem of this paper.

*Proof of Theorem 1.1.* Recall that  $\widehat{\Phi} = \psi \circ \widetilde{\Phi} \circ \psi'$ . By Proposition 4.1,  $\widetilde{\Phi}$  preserves the grading. Moreover, it follows from our constructions in §2.3 and §3.5 that  $\psi'$  and  $\psi$  preserve the grading. Therefore  $\widehat{\Phi}$  preserves the grading.

Now, the maps  $\iota_*$  in (1) preserve the grading. Therefore since the diagram (1) commutes, the isomorphism  $\Phi$  also preserves the grading, i.e.  $\Phi$  maps  $HF_\rho^+(-Y)$  to  $ECH_\rho(Y)$  for all  $\rho \in \text{Vect}(Y)$ .  $\square$

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